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The First Exit Time Problem

Christos H. Skiadas and Charilaos Skiadas

Data Analysis and Forecasting Laboratory, Technical University of Crete
Chania, Crete, Greece

Email: skiadas@smta.net

Hanover College, Hanover, Indiana, USA

Email: skiadas@hanover.edu

Abstract: In this paper we present and analyze the first exit time problem. The main parts of the underlying stochastic theory are given. Stochastic simulations take place and illustrations of the probability density functions for few cases are presented.

Keywords: Stochastic modeling, The first exit time problem, Linear boundary, Curved boundary, Fokker-Planck equation, Stochastic differential equation

1 Introduction

The first exit time distribution for a stochastic process is the distribution of the times at which particles following this process cross a certain (often linear) barrier. It is often referred to also as hitting time. It is closely related to the probability density function $p(x_t, t)$ of a stochastic process x_t over time t .

For a linear horizontal barrier located at a , the first exit time density function relation is given by:

$$g(t) = \frac{|a|}{t} p(a, t) .$$

For other types barriers (e.g. quadratic), a tangent approximation may be used to obtain a satisfactory estimate as is presented below.

The probability density function may be computed in some cases using the Fokker-Planck equation. In particular in the one-dimensional diffusion problem expressed by a stochastic differential equation of the form:

$$dx_t = \sigma dw_t ,$$

where σ is the variance and w_t is the standard Wiener process, the corresponding Fokker-Planck equation for the probability density function $p(x_t, t)$ associated to the above stochastic differential equation has the form:

$$\frac{\partial p(x_t, t)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p(x_t)}{\partial x^2} .$$

This partial differential equation form is also known as the one-dimensional heat equation first solved by Joseph Fourier (1822). Later on Fick (1855a, b) applied this equation to express one-dimensional diffusion in solids.

Albert Einstein (1905) proposed the same form for the one-dimensional diffusion for solving the Brownian motion process. It was the first derivation and application of a probabilistic-stochastic theory to the classical Brownian motion problem that is the movement of a particle or a molecule into a liquid. He resulted in giving the development over space and time of this particle. One year later Smoluchowski (1906) proposed also a theory for solving the Brownian motion problem.

Solving this partial differential equation with the boundary conditions,

$$p(x_t, 0 : 0, 0) = \delta(x_t, 0) \text{ and } \frac{\partial p(x_t, t : 0, t)}{\partial x} = 0 \text{ as } x_t \rightarrow \infty ,$$

the probability density function p_t for the stochastic process results:

$$p(x_t, t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{x_t^2}{2\sigma^2 t}} .$$

2 The First Exit Time Density Function

The finding of a density function expressing the distribution of the first exit time of particles escaping from a boundary is due to Schrödinger (1915) and Smoluchowsky (1915) in two papers published in the same journal issue. Later on Siegert (1951) gave an interpretation closer to our modern notation whereas Jennen (1985), Lerche (1986) and and Jennen and Lerche (1981) gave the most interesting first exit density function form. For the simple case presented earlier the proposed form is:

$$g(t) = \frac{|a|}{t} p(a, t) = \frac{|a|}{\sigma\sqrt{2\pi^3}} e^{-\frac{a^2}{2\sigma^2 t}}$$

Jennen (1985) proposed a more general form using a tangent approximation of the first exit density. Application of this theory to the mortality modeling leads to the following form (earlier work can be found in Janssen and Skiadas (1995) and Skiadas and Skiadas (2007, 2010):

$$g(t) = \frac{|(H_t - tH_t')|}{t} p(t) = \frac{|H_t - tH_t'|}{\sigma\sqrt{2\pi^3}} e^{-\frac{(H_t)^2}{2\sigma^2 t}} .$$

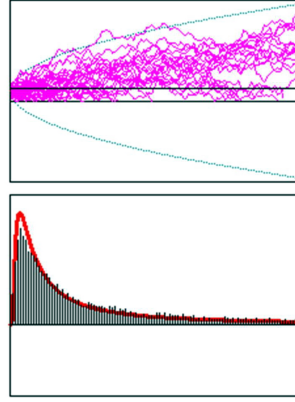


Fig. 1. Linear barrier

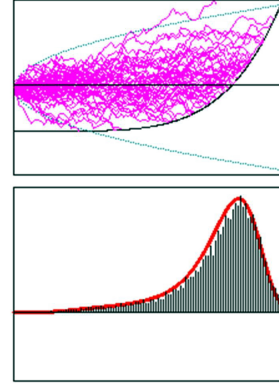


Fig. 2. Curved barrier

The last form is associated to the following stochastic process:

$$dx_t = \mu_t dt + \sigma dw_t$$

where μ_t is a function of time and there exists a function H_t related to μ_t with the differential equation: $\mu_t = dH / dt$

The associated Fokker-Planck equation is:

$$\frac{\partial p(x_t, t)}{\partial t} = -\mu_t \frac{\partial p(x_t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(x_t)}{\partial x^2},$$

and the solution is given by:

$$p(t) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(H_t)^2}{2\sigma^2 t}}.$$

Two realizations are provided in figures 1 and 2. In the first case the first exit time probability density function is provided and stochastic simulations are done for a linear barrier located at a . figure 2 illustrates the case when a curved barrier is present.

3 Conclusions

We have briefly discussed the first exit time problem; first giving a definition and then presenting the main findings. Stochastic simulations are done and the related graphs are presented. Several references related to the key-points of the stochastic process are included.

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